Problem T3. Protostar formation *(9 points)*

i. (0.8 pts)

$$T = \text{const} \implies pV = \text{const}$$
$$V \propto r^3$$
$$\therefore p \propto r^{-3} \implies \frac{p(r_1)}{p(r_0)} = 2^3 = 8.$$

ii. (1 pt) During the period considered the pressure is negligible. Therefore the gas is in free fall. By Gauss' theorem and symmetry, the gravitational field at any point in the ball is equivalent to the one generated when all the mass closer to the center is compressed into the center. Moreover, while the ball has not yet shrunk much, the field strength on its surface does not change much either. The acceleration of the outermost layer stays approximately constant. Thus,

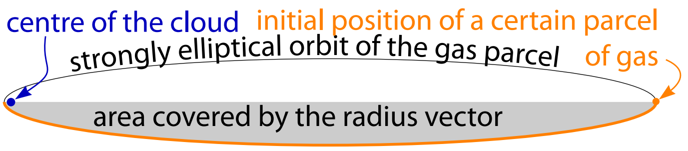
$$t \approx \sqrt{\frac{2(r_0 - r_2)}{g}}$$

where

$$g \approx \frac{Gm}{r_0^2},$$
$$\therefore t \approx \sqrt{\frac{2r_0^2(r_0 - r_2)}{Gm}} = \sqrt{\frac{0.1 r_0^3}{Gm}}.$$

iii. (2.5 pts) Gravitationally the outer layer of the ball is influenced by the rest just as the rest were compressed into a point mass. Therefore we have Keplerian motion: the fall of any part of the outer layer consists in a halfperiod of an ultra-elliptical orbit. The ellipse is degenerate into a line; its foci are at the ends of the line; one focus is at the center of the ball (by Kepler's 1st law) and the other one is at r_0, see figure (instead of a degenerate ellipse, a strongly elliptical ellipse is depicted). The period of the orbit is determined by the longer semiaxis of the ellipse (by Kepler's 3rd law). The longer semiaxis is $r_0/2$ and we are interested in half a period. Thus, the answer is equal to the halfperiod of a circular orbit of radius $r_0/2$:

$$\left(\frac{2\pi}{2t_{r\to 0}}\right)^2 \frac{r_0}{2} = \frac{Gm}{(r_0/2)^2} \implies t_{r\to 0} = \pi\sqrt{\frac{r_0^3}{8Gm}}.$$



Alternatively, one may write the energy conservation law $\frac{\dot r^2}{2} - \frac{Gm}{r} = E$ (that in turn is obtainable from Newton's II law $\ddot r = -\frac{Gm}{r^2}$) with $E = -\frac{Gm}{r_0}$, separate the variables ($\frac{dr}{dt} = -\sqrt{2E + \frac{2Gm}{r}}$) and write the integral $t = -\int \frac{dr}{\sqrt{2E+\frac{2Gm}{r}}}$. This integral is probably not calculable during the limitted time given during the Olympiad, but a possible approach can be sketched as follows. Substituting $\sqrt{2E + \frac{2Gm}{r}} = \xi$ and $\sqrt{2E} = v$, one gets

$$\frac{t_\infty}{4Gm} = \int_0^\infty \frac{d\xi}{(v^2 - \xi^2)^2}$$
$$= \frac{1}{4v^3}\int_0^\infty \left[\frac{v}{(v-\xi)^2} + \frac{v}{(v+\xi)^2} + \frac{1}{v-\xi} + \frac{1}{v+\xi}\right] d\xi.$$

Here (after shifting the variable) one can use $\int \frac{d\xi}{\xi} = \ln\xi$ and $\int \frac{d\xi}{\xi^2} = -\frac{1}{\xi}$, finally getting the same answer as by Kepler's laws.

iv. (1.7 pts) By Clapeyron–Mendeleyev law,

$$p = \frac{mRT_0}{\mu V}.$$

Work done by gravity to compress the ball is

$$W = -\int p\,dV = -\frac{mRT_0}{\mu}\int_{\frac{4}{3}\pi r_0^3}^{\frac{4}{3}\pi r_3^3} \frac{dV}{V} = \frac{3mRT_0}{\mu}\ln\frac{r_0}{r_3}.$$

The temperature stays constant, so the internal energy does not change; hence, according to the 1st law of thermodynamics, the compression work W is the heat radiated.

v. (1 pt) The collapse continues adiabatically.

$$pV^\gamma = \text{const} \implies TV^{\gamma-1} = \text{const}.$$
$$\therefore T \propto V^{1-\gamma} \propto r^{3-3\gamma}$$
$$\therefore T = T_0\left(\frac{r_3}{r}\right)^{3\gamma-3}.$$

vi. (2 pts) During the collapse, the gravitational energy is converted into heat. Since $r_3 \gg r_4$, The released gravitational energy can be estimated as $\Delta\Pi = -Gm^2(r_4^{-1} - r_3^{-1}) \approx -Gm^2/r_4$ (exact calculation by integration adds a prefactor $\frac{3}{5}$); the terminal heat energy is estimated as $\Delta Q = c_V\frac{m}{\mu}(T_4 - T_0) \approx c_V\frac{m}{\mu}T_4$ (the approximation $T_4 \gg T_0$ follows from the result of the previous question, when combined with $r_3 \gg r_4$). So, $\Delta Q = \frac{R}{\gamma-1}\frac{m}{\mu}T_4 \approx \frac{m}{\mu}RT_4$. For the temperature T_4, we can use the result of the previous question, $T_4 = T_0\left(\frac{r_3}{r_4}\right)^{3\gamma-3}$. Since initial full energy was approximately zero, $\Delta Q + \Delta\Pi \approx 0$, we obtain

$$\frac{Gm^2}{r_4} \approx \frac{m}{\mu}RT_0\left(\frac{r_3}{r_4}\right)^{3\gamma-3} \implies r_4 \approx r_3\left(\frac{RT_0 r_3}{\mu m G}\right)^{\frac{1}{3\gamma-4}}.$$

Therefore,

$$T_4 \approx T_0\left(\frac{RT_0 r_3}{\mu m G}\right)^{\frac{3\gamma-3}{4-3\gamma}}.$$

Alternatively, one can obtain the result by approximately equating the hydrostatic pressure $\rho r_4 \frac{Gm}{r_4^2}$ to the gas pressure $p_4 = \frac{\rho}{\mu}RT_4$; the result will be exactly the same as given above.