Problem T2. Kelvin water dropper *(8 points)*

Part A. Single pipe *(4 points)*

i. (1.2 pts) Let us write the force balance for the droplet. Since $d \ll r$, we can neglect the force $\frac{\pi}{4}\Delta p d^2$ due to the excess pressure Δp inside the tube. So, the gravity force $\frac{4}{3}\pi r_{\max}^3 \rho g$ is balanced by the capillary force. When the droplet separates from the tube, the water surface forms in the vicinity of the nozzle a "neck", which has vertical tangent. In the horizontal cross-section of that "neck", the capillary force is vertical and can be calculated as $\pi\sigma d$. So,

$$r_{\max} = \sqrt[3]{\frac{3\sigma d}{4\rho g}}.$$

ii. (1.2 pts) Since $d \ll r$, we can neglect the change of the droplet's capacitance due to the tube. On the one hand, the droplet's potential is φ; on the other hand, it is $\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$. So,

$$Q = 4\pi\varepsilon_0\varphi r.$$

iii. (1.6 pts) Excess pressure inside the droplet is caused by the capillary pressure $2\sigma/r$ (increases the inside pressure), and by the electrostatic pressure $\frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0\varphi^2/r^2$ (decreases the pressure). So, the sign of the excess pressure will change, if $\frac{1}{2}\varepsilon_0\varphi_{\max}^2/r^2 = 2\sigma/r$, hence

$$\varphi_{\max} = 2\sqrt{\sigma r/\varepsilon_0}.$$

The expression for the electrostatic pressure used above can be derived as follows. The electrostatic force acting on a surface charge of density σ and surface area S is given by $F = \sigma S \cdot \bar{E}$, where $\bar{E}$ is the field at the site without the field created by the surface charge element itself. Note that this force is perpendicular to the surface, so F/S can be interpreted as a pressure. The surface charge gives rise to a field drop on the surface equal to $\Delta E = \sigma/\varepsilon_0$ (which follows from the Gauss law); inside the droplet, there is no field due to the conductivity of the droplet: $\bar{E} - \frac{1}{2}\Delta E = 0$; outside the droplet, there is field $E = \bar{E} + \frac{1}{2}\Delta E$, therefore $\bar{E} = \frac{1}{2}E = \frac{1}{2}\Delta E$. Bringing everything together, we obtain the expression used above.

Note that alternatively, this expression can be derived by considering a virtual displacement of a capacitor's surface and comparing the pressure work $p\Delta V$ with the change of the electrostatic field energy $\frac{1}{2}\varepsilon_0 E^2 \Delta V$.

Finally, the answer to the question can be also derived from the requirement that the mechanical work dA done for an infinitesimal droplet inflation needs to be zero. From the energy conservation law, $dW + dW_{\text{el}} = \sigma\, d(4\pi r^2) + \frac{1}{2}\varphi_{\max}^2\, dC_d$, where the droplet's capacitance $C_d = 4\pi\varepsilon_0 r$; the electrical work $dW_{\text{el}} = \varphi_{\max} dq = 4\pi\varepsilon_0\varphi_{\max}^2 dr$. Putting $dW = 0$ we obtain an equation for $\varphi_{\max}$, which recovers the earlier result.

Part B. Two pipes *(4 points)*

i. (1.2 pts) This is basically the same as Part A-ii, except that the surroundings' potential is that of the surrounding electrode, $-U/2$ (where $U = q/C$ is the capacitor's voltage) and droplet has the ground potential (0). As it is not defined which electrode is the positive one, opposite sign of the potential may be chosen, if done consistently. Note that since the cylindrical electrode is long, it shields effectively the environment's (ground, wall, etc) potential. So, relative to its surroundings, the droplet's potential is $U/2$. Using the result of Part A we obtain

$$Q = 2\pi\varepsilon_0 U r_{\max} = 2\pi\varepsilon_0 q r_{\max}/C.$$

ii. (1.5 pts) The sign of the droplet's charge is the same as that of the capacitor's opposite plate (which is connected to the farther electrode). So, when the droplet falls into the bowl, it will increase the capacitor's charge by Q:

$$dq = 2\pi\varepsilon_0 U r_{\max} dN = 2\pi\varepsilon_0 r_{\max} n dt \frac{q}{C},$$

where $dN = ndt$ is the number of droplets which fall during the time dt This is a simple linear differential equation which is solved easily to obtain

$$q = q_0 e^{\gamma t}, \quad \gamma = \frac{2\pi\varepsilon_0 r_{\max} n}{C} = \frac{\pi\varepsilon_0 n}{C}\sqrt[3]{\frac{6\sigma d}{\rho g}}.$$

iii. (1.3 pts) The droplets can reach the bowls if their mechanical energy mgH (where m is the droplet's mass) is large enough to overcome the electrostatic push: The droplet starts at the point where the electric potential is 0, which is the sum of the potential $U/2$, due to the electrode, and of its self-generated potential $-U/2$. Its motion is not affected by the self-generated field, so it needs to fall from the potential $U/2$ down to the potential $-U/2$, resulting in the change of the electrostatic energy equal to $UQ \le mgH$, where $Q = 2\pi\varepsilon_0 U r_{\max}$ (see above). So,

$$U_{\max} = \frac{mgH}{2\pi\varepsilon_0 U_{\max} r_{\max}},$$

$$\therefore U_{\max} = \sqrt{\frac{H\sigma d}{2\varepsilon_0 r_{\max}}} = \sqrt[6]{\frac{H^3 g\sigma^2\rho d^2}{6\varepsilon_0^3}}.$$